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Fundamental natural frequencies of double-walled carbon nanotubes

Isaac Elishakoff*, Demetris Pentaras

Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0991, USA

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Abstract

This study deals with evaluation of fundamental natural frequencies of double-walled carbon nanotubes under various boundary conditions. The Bubnov–Galerkin and Petrov–Galerkin methods are applied to derive the expressions for natural frequencies. Apparently for the first time in the literature explicit expressions are obtained for the natural frequencies. These can be useful for the designer to estimate the fundamental frequency in each of two series.
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1. Introduction

As Qian et al. [1] mention that "the discovery of multi-walled carbon nanotubes (MWCNTs) in 1991 has stimulated ever-broader research activities in science and engineering devoted entirely to carbon nanostructures and their applications. This is due in large part to the combination of their expected structural perfection, small size, low density, high stiffness, high strength (the tensile strength of the outer most shell of MWCNT is approximately 100 time greater that that of aluminum), and excellent electronic properties. As a result, carbon nanotubes (CNT) may find use in a wide range of application in material reinforcement, field emission pane display, chemical sensing, drug delivery, and nanoelectronics." The state of the art in modeling and simulation of nanostructured materials and systems was given by Gates and Hinkley [2] and Liu et al. [3] inter alia. Vibrations of double-walled carbon nanotubes (DWCNTs) have been considered in several papers. Xu et al. [4,5] and Ru [6] studied the free vibrations of a DWCNT which composed of two coaxial single-walled CNT interacting each other by the interlayer van der Waals forces. Therefore, the inner and outer CNT are modeled as two individual elastic beams [4–6]. In these studies the Euler-beam model has been used to derive exact solution for the natural frequencies at various boundary conditions. The results showed that DWCNTs have frequencies in the range of terahertz (THz). Also, in the study of vibration of CNT, Timoshenko beam model has been used for short length-to-diameter ratios which allows for the effect of transverse shear deformation [7,8]. Likewise, the shell models have been applied recently by He et al. [9], Ru [10] and Wang et al. [11]. Ru [6] stresses that "carbon MWCNTs are different from

*Corresponding author. Tel.: +1 561 297 3457; fax: +1 561 297 2825.
E-mail addresses: elishako@fau.edu (I. Elishakoff), dpentara@fau.edu (D. Pentaras).

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traditional elastic beams due to their hollow multilayer structure and the associated interlayer van der Waals forces." The exact solutions lead to the need of solving transcendental equations. It appears that they can be usefully supplemented by simple solutions.

In this paper, approximate solutions are found by using Bubnov–Galerkin [12,13] and Petrov–Galerkin [14] methods. Explicit formulas of natural frequencies are derived for the DWCNTs at different boundary conditions. Comparison of the results with recent studies shows that the above methods constitute effective alternative techniques to exact solution for studying the vibration properties of CNT.

2. Analysis

The governing differential equations for free vibration of the DWCNTs read

$$c_1(w_2 - w_1) = EI_1\frac{\partial^4 w_1}{\partial x^4} + \rho A_1 \frac{\partial^2 w_1}{\partial t^2}$$

$$-c_1(w_2 - w_1) = EI_2\frac{\partial^4 w_2}{\partial x^4} + \rho A_2 \frac{\partial^2 w_2}{\partial t^2} \tag{1}$$

where x is the axial coordinate, t the time, $w_j(x, t)$ the transverse displacement, I_j the moment of inertia and A_j the cross-sectional area of the jth nanotube; the indexes $j = 1,2$ denote the inner tube and outer tube, respectively.

The exact solutions for various boundary conditions were considered by Xu et al. [4,5]. Their derivation necessitates numerical evaluation of 8×8 determinant and attendant cumbersome numerical analysis. Therefore, it appears imperative to obtain explicit expressions for natural frequencies by approximate methods. Here, we utilize the Bubnov–Galerkin and Petrov–Galerkin methods.

3. Simply supported DWCNT: exact solution

For the DWCNT that is simply supported at both ends one obtains an exact solution by substitution

$$w_1 = Y_1 \sin(m\pi\xi)\sin(\omega t)$$

$$w_2 = Y_2 \sin(m\pi\xi)\sin(\omega t) \tag{2}$$

where $\xi = x/L$ is a non-dimensional axial coordinate and $m = 1,2,\ldots$ the number of half-waves in the longitudinal direction as well as the sequence number of the vibrational mode and ω the sought natural frequency. We substitute Eq. (2) into Eq. (1) and demand nontriviality of Y_1 and/or Y_2. In order $Y_1^2 + Y_2^2$ to be different from zero the following determinant must vanish

$$\begin{vmatrix} EI_1(m\pi/L)^4 - \rho A_1\omega^2 + c_1 & -c_1 \\ c_1 & -EI_2(m\pi/L)^4 + \rho A_2\omega^2 - c_1 \end{vmatrix} \tag{3}$$

The equation for natural frequency squared ω^2 reads

$$\rho^2 A_1 A_2 \omega^4 - (A_1 c_1 L^4 + A_1 EI_2 m^4\pi^4 + EI_1 m^4 \pi^4 A_2 + c_1 A_2 L^4 \rho)\omega^2/L^4 (EI_1 m^4\pi^4 c_1 L^4 + E^2 I_1 m^8 \pi^8 I_2 + c_1 L^4 EI_2 m^4 \pi^4)/L^8 = 0 \tag{4}$$

By letting $m = 1$ and solving Eq. (4), we get the following solutions for ω^2:

$$\omega_{1,1}^2 = \tfrac{1}{2}[L^4 A_1 c_1 + L^4 c_1 A_2 + A_1 EI_2 \pi^4 + EI_1 A_2 \pi^4 - (L^8 A_1^2 c_1^2 + 2L^8 A_1 c_1^2 A_2 + 2L^4 A_1^2 c_1 EI_2 \pi^4 - L^4 A_1 c_1 EI_1 A_2 \pi^4 + L^8 c_1^2 A_2^2 - 2L^4 c_1 A_2 A_1 EI_2 \pi^4 + 2L^4 c_1 A_2^2 EI_1 \pi^4 + A_1^2 E^2 I_2^2 \pi^8 - 2A_1 E^2 I_2 I_1 A_2 \pi^8 + \tfrac{7873}{3319} E^2 I_1^2 A_2^2 \pi^8)^{1/2}]/L^4 \rho A_1 A_2$$

Table 1
First natural frequencies in each series for various values of L/d (simply supported DWCNTs at both ends).

L/d	10	11	12	13	14	15	16	17	18	19	20
$\omega_{1,1}$ (10^{12} Hz)	0.46830	0.38707	0.32527	0.27716	0.23899	0.20819	0.18298	0.16209	0.14458	0.12976	0.11711
$\omega_{2,1}$ (10^{12} Hz)	7.7721	7.7690	7.7670	7.7657	7.7648	7.7642	7.7638	7.7634	7.7632	7.7630	7.7629

$$\omega_{2,1}^2 = \tfrac{1}{2}[L^4A_1c_1 + L^4c_1A_2 + A_1EI_2\pi^4 + EI_1A_2\pi^4 + (L^8A_1^2c_1^2 + 2L^8A_1c_1^2A_2 + 2L^4A_1^2c_1EI_2\pi^4 - L^4A_1c_1EI_1A_2\pi^4 + L^8c_1^2A_2^2 - 2L^4c_1A_2A_1EI_2\pi^4 + 2L^4c_1A_2^2EI_1\pi^4 + A_1^2E^2I_2^2\pi^8 - 2A_1E^2I_2I_1A_2\pi^8 + \tfrac{7873}{3319}E^2I_1^2A_2^2\pi^8)^{1/2}]/L^4\rho A_1A_2 \quad (5)$$

The indexes 1 and 2 indicate the first and second series of frequencies, respectively. For numerical analysis hereinafter we fix the values of the Young's modulus E at 1 TPa and the mass density $\rho = 2.3\,\mathrm{g/cm^3}$ following Yoon et al. [7] and Xu et al. [4]. The van der Waals interlayer interaction coefficient is fixed at $c_1 = 71.11$ GPa [4]; inner radius R_1 equals 0.35 nm, whereas the outer radius R_2 equals 0.7 nm. Table 1 lists the first natural frequency in each series for various values of L/d.

The value for $\omega_{1,1} = 0.4683$ evaluated for $L/d = 10$ is close to the value 0.46 THz reported by Xu et al. [4]. The first natural frequency in the second series $\omega_{2,1} = 7.7721$ is close to the value $\omega_{2,1} = 7.71$ THz reported by Xu et al. [4]. Present value is 0.78 percent above Xu's result. For $L/d = 20$, Xu et al. [4] report the values $\omega_{1,1} = 0.11$ THz and $\omega_{2,1} = 7.76$ THz, respectively. These correlate well with our values of 0.12 and 7.76 THz, respectively.

The exact solution given in Eqs. (4) and (5) will then serve as the benchmark solution, against which the efficacy of approximate solutions will be tested.

4. Simply supported DWCNT: Bubnov–Galerkin method

In order to ascertain the accuracy of the approximate solutions we first apply the Bubnov–Galerkin [12,13] method to the simply supported DWCNT the exact solution for which was reported in preceding section.

We approximate the displacements as follows:

$$w_1 = D_1\varphi^{(1)}, \quad w_2 = D_2\varphi^{(1)} \quad (6)$$

where

$$\varphi^{(1)} = 3\xi^5 - 10\xi^3 + 7\xi \quad (7)$$

The function in Eq. (7) is so called Duncan polynomial [15]. We substitute the expressions (6) into governing differential Eq. (1) having in mind Eq. (2); we then multiply the result of the substitution by $\varphi^{(1)}(\xi)$ and integrate over the length of the beam. The natural frequencies derived via approximate methods are denoted by overbar, as $\overline{\omega}$. We get the following two equations for D_1 and D_2:

$$(-L^4\rho A_1\overline{\omega}^2 + L^4c_1 + 99EI_1)D_1 + (-L^4c_1D_2) = 0$$

$$-L^4c_1D_1 + (-L^4\rho A_2\overline{\omega}^2 + L^4c_1 + 99EI_2)D_2 = 0 \quad (8)$$

We demand the determinant

$$\begin{vmatrix} -L^4\rho A_1\overline{\omega}^2 + L^4c_1 + 99EI_1 & -L^4c_1 \\ -L^4c_1 & -L^4\rho A_2\overline{\omega}^2 + L^4c_1 + 99EI_2 \end{vmatrix} \quad (9)$$

to vanish. This leads to the frequency equation

$$L^8\rho^2A_1A_2\overline{\omega}^4 + (99L^4A_1EI_2 - L^8A_1c_1 - 99EI_1L^4A_2 - L^8c_1A_2)\rho\overline{\omega}^2 + 99L^4c_1EI_2 + 99EI_1L^4c_1 + 9801E^2I_1I_2 = 0 \quad (10)$$

with roots

$$\overline{\omega}_{1,1}^2 = [L^4A_1c_1 + L^4c_1A_2 + 99A_1EI_2 + 99EI_1A_2 - (L^8A_1^2c_1^2 + 2L^8A_1c_1^2A_2 + 198L^4A_1^2c_1EI_2 - 198L^4A_1c_1EI_1A_2 + L^8c_1^2A_2^2 - 198L^4c_1A_2A_1EI_2 + 198L^4c_1A_2^2EI_1 + 9801A_1^2E^2I_2^2 - 19\,602A_1E^2I_2I_1A_2 + 9801E^2I_1^2A_2^2)^{1/2}]/2L^4\rho A_1A_2$$

$$\overline{\omega}_{2,1}^2 = [L^4A_1c_1 + L^4c_1A_2 + 99A_1EI_2 + 99EI_1A_2 + (L^8A_1^2c_1^2 + 2L^8A_1c_1^2A_2 + 198L^4A_1^2c_1EI_2 - 198L^4A_1c_1EI_1A_2 + L^8c_1^2A_2^2 - 198L^4c_1A_2A_1EI_2 + 198L^4c_1A_2^2EI_1 + 9801A_1^2E^2I_2^2 - 19602A_1E^2I_2I_1A_2 + 9801E^2I_1^2A_2^2)^{1/2}]/2L^4\rho A_1A_2 \tag{11}$$

For the data adopted in the Section 3, for $L/d = 10$ we obtain from Eq. (11) the following frequencies:

$$\overline{\omega}_{1,1} = 0.4721064\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.7722259\,\text{THz} \tag{12}$$

The percentagewise difference between $\overline{\omega}_{1,1}$ and the exact value in Table 1 $\omega_{1,1}$ is 0.81 percent, whereas the respective difference between $\overline{\omega}_{2,1}$ and $\omega_{2,1}$ is 0.0013 percent. This demonstrates the high accuracy of the Bubnov–Galerkin method for the DWCNTs.

Consider now another coordinate function, namely

$$\varphi^{(2)} = \xi^4 - 2\xi^3 + \xi \tag{13}$$

instead of $\varphi^{(1)}$ in Eq. (7). This function is due to Duncan [15] although in completely different context. The procedure as described above leads to frequency determinant

$$\begin{vmatrix} -31L^4\rho A_1\overline{\omega}^2 + 31L^4c_1 + 3024EI_1 & -31L^4c_1 \\ -31L^4c_1 & -31L^4\rho A_2\overline{\omega}^2 + 31L^4c_1 + 3024EI_2 \end{vmatrix} \tag{14}$$

and attendant frequency equation

$$961L^8\rho^2A_1A_2\bar{\omega}^4 - (93\,744L^4A_1EI_2 + 961L^8A_1c_1 + 93\,744EI_1L^4A_2 + 961L^8c_1A_2)\rho\overline{\omega}^2 + 93\,744L^4c_1EI_2 + 93\,744EI_1L^4c_1 + 9\,144\,576E^2I_1I_2 = 0 \tag{15}$$

whose roots are

$$\overline{\omega}_{1,1}^2 = [31L^4A_1c_1 + 31L^4c_1A_2 + 3024A_1EI_2 + 3024EI_1A_2 - (961L^8A_1^2c_1^2 + 1922L^8A_1c_1^2A_2 + 187\,488L^4A_1^22c_1EI_2 - 187\,488L^4A_1c_1EI_1A_2 + 961L^8c_1^2A_2^2 - 187\,488L^4c_1A_2A_1EI_2 + 187\,488L^4c_1A_2^2EI_1 + 9\,144\,576A_1^2E^2I_2^2 - 18\,289\,152A_1E^2I_2I_1A_2 + 9\,144\,576E^2I_1^2A_2^2)^{1/2}]/62L^4\rho A_1A_2$$

$$\overline{\omega}_{2,1}^2 = 31L^4A_1c_1 + 31L^4c_1A_2 + 3024A_1EI_2 + 3024EI_1A_2 + (961L^8A_1^2c_1^2 + 1922L^8A_1c_1^2A_2 + 187\,488L^4A_1^22c_1EI_2 - 187\,488L^4A_1c_1EI_1A_2 + 961L^8c_1^2A_2^2 - 187\,488L^4c_1A_2A_1EI_2 + 187\,488L^4c_1A_2^2EI_1 + 9\,144\,576A_1^2E^2I_2^2 - 18\,289\,152A_1E^2I_2I_1A_2 + 9\,144\,576E^2I_1^2A_2^2)^{1/2}]/62L^4\rho A_1A_2 \tag{16}$$

The numerical values for the data listed in Section 3, for $L/D = 10$ we obtain from Eq. (16) the following natural frequencies

$$\overline{\omega}_{1,1} = 0.4686349\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.7720800\,\text{THz} \tag{17}$$

It is seen that the coordinate function $\varphi^{(2)}$ in Eq. (13) leads to lower value of the fundamental frequency estimate than use of the function $\varphi^{(1)}$ in Eq. (7). Therefore, the expressions in Eq. (16) are preferable to those in Eq. (11). Additionally, the percentagewise difference with the exact solution in estimation of $\overline{\omega}_{1,1}$ constitutes 0.07 percent, whereas its counterpart for $\overline{\omega}_{2,1}$ is 0.0003 percent. These small differences attest for the high reliability of the Bubnov–Galerkin method for studying DWCNTs.

5. Simply-supported DWCNTs: Petrov–Galerkin Method

In 1940 Petrov [14] suggested a modification to the Bubnov–Galerkin method; he proposed to employ two systems of functions simultaneously; namely, one system of functions is used to approximate the displacement, whereas another set of functions is used for satisfying the orthogonality condition.

We first substitute the coordinate function $\varphi^{(2)}$ in Eq. (13) into governing equations as was done in Eq. (6). The results however are not multiplied by $\varphi^{(2)}$ as in Bubnov–Galerkin method, but by some *other* function. We choose this multiplicative function to be

$$\psi = \xi^6 - 5\xi^3 + 4\xi \tag{18}$$

Performing the Petrov–Galerkin procedure results in the following equations for D_1 and D_2:

$$(169L^4\rho A_1\overline{\omega}^2 - 169L^4c_1 - 16\,500EI_1)D_1 + 169L^4c_1D_2 = 0$$

$$169L^4c_1D_1 + (169L^4\rho A_2\overline{\omega}^2 - 169L^4c_1 - 16\,500EI_2)D_2 = 0 \tag{19}$$

The frequency equation reads

$$\begin{aligned}&28\,561L^8\rho^2A_1A_2\bar{\omega}^4 + (-2\,788\,500L^4A_1EI_2 - 28\,561L^8A_1c_1 - 2\,788\,500EI_1L^4A_2 - 28\,561L^8c_1A_2)\rho\overline{\omega}^2\\&\quad + 28\,561L^4c_1EI_2 + 2\,788\,500EI_1L^4c_1 + 272\,250\,000E^2I_1I_2 = 0\end{aligned} \tag{20}$$

with roots

$$\begin{aligned}\overline{\omega}_{1,1}^2 = {}&[169L^4A_1c_1 + 169L^4c_1A_2 + 16\,500A_1EI_2 + 16\,500EI_1A_2 - (28\,561L^8A_1^2c_1^2 + 57\,122L^8A_1c_1^2A_2\\&+ 5\,577\,000L^4A_1^2c_1EI_2 - 5\,577\,000L^4A_1c_1EI_1A_2 + 28\,561L^8c_1^2A_2^2 - 5\,577\,000L^4c_1A_2A_1EI_2\\&+ 5\,577\,000L^4c_1A_2^2EI_1 + 272\,250\,000A_1^2E^2I_2^2 - 544\,500\,000A_1E^2I_2I_1A_2\\&+ 272\,250\,000E^2I_1^2A_2^2)^{1/2}]/338L^4\rho A_1A_2\end{aligned}$$

$$\begin{aligned}\overline{\omega}_{2,1}^2 = {}&[169L^4A_1c_1 + 169L^4c_1A_2 + 16\,500A_1EI_2 + 16\,500EI_1A_2 - +(28\,561L^8A_1^2c_1^2 + 57\,122L^8A_1c_1^2A_2\\&+ 5\,577\,000L^4A_1^2c_1EI_2 - 5\,577\,000L^4A_1c_1EI_1A_2 + 28\,561L^8c_1^2A_2^2 - 5\,577\,000L^4c_1A_2A_1EI_2\\&+ 5\,577\,000L^4c_1A_2^2EI_1 + 272\,250\,000A_1^2E^2I_2^2 - 544\,500\,000A_1E^2I_2I_1A_2\\&+ 272\,250\,000E^2I_1^2A_2^2)^{1/2}]/338L^4\rho A_1A_2\end{aligned} \tag{21}$$

Numerical evaluation of Eq. (21) yields for $L/d = 10$,

$$\overline{\omega}_{1,1} = 0.4688382\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.7720885\,\text{THz} \tag{22}$$

Comparison of the results in Eq. (22) with the values in Eq. (17) show that $\overline{\omega}_{1,1}$ obtained by Petrov–Galerkin method is slightly higher than that given by Bubnov–Galerkin method; however, the estimate for $\overline{\omega}_{2,1}$ resulting from both the Petrov–Galerkin and Bubnov–Galerkin methods are nearly coincident.

Likewise, it is remarkable that the combinations of pair of functions

$$\varphi^{(2)} = \xi^4 - 2\xi^3 + \xi, \quad \psi = 3\xi^5 - 10\xi^3 + 7\xi \tag{23}$$

or the one obtained by the reversal of the substitution and the multiplication functions

$$\varphi^{(1)} = 3\xi^5 - 10\xi^3 + 7\xi, \quad \psi = \xi^4 - 2\xi^3 + \xi \tag{24}$$

both yield the same results as the Bubnov–Galerkin method where the coordinate function $\varphi^{(2)}$ in Eq. (13) is used.

It must be also noted that the use of the combination

$$\varphi^{(3)} = \sin(\pi\xi), \quad \psi = \xi^4 - 2\xi^3 + \xi \tag{25}$$

or

$$\varphi^{(2)} = \xi^4 - 2\xi^3 + \xi, \quad \psi = \sin(\pi\xi) \tag{26}$$

both yield exact expressions of the natural frequency in Eq. (5). This result must have been anticipated since in Eqs. (25) and (26), either the substitution function φ or the multiplicative function ψ coincides with the exact mode shape.

6. Clamped–clamped DWCNT: Bubnov–Galerkin Method

For this set of boundary conditions, we use so called Filonenko–Borodich [16] trigonometric polynomial as a coordinate function in the context of Bubnov–Galerkin method:

$$\varphi^{(4)} = 1 - \cos(2\pi\xi) \tag{27}$$

The usual Bubnov–Galerkin procedure yields the equations for D_1 and D_2:

$$(3L^4\rho A_1\overline{\omega}^2 - 3L^4c_1 - 16\pi^4EI_1)D_1 + 3L^4c_1D_2 = 0$$

$$3L^4c_1D_1 + (3L^4\rho A_1\overline{\omega}^2 - 3L^4c_1 - 16\pi^4EI_2)D_2 = 0 \tag{28}$$

We arrive in the following frequency equation:

$$9L^8\rho^2A_1A_2\overline{\omega}^4 + (-\tfrac{23\,378}{5}L^4A_1EI_2 - 9L^8A_1c_1 - \tfrac{23\,378}{5}EI_1L^4A_2 - 9L^8c_1A_2)\rho\overline{\omega}^2 + \tfrac{23378}{5}L^4c_1EI_2 + \tfrac{23\,378}{5}EI_1L^4c_1 + 2\,429\,100E^2I_1I_2 = 0 \tag{29}$$

with roots

$$\overline{\omega}_{1,1}^2 = [\tfrac{1}{2}L^4A_1c_1 + \tfrac{1}{2}L^4c_1A_2 + \tfrac{8572}{33}A_1EI_2 + \tfrac{8572}{33}EI_1A_2 - (\tfrac{1}{4}L^8A_1^2c_1^2 + \tfrac{1}{2}L^8A_1c_1^2A_2 + \tfrac{8572}{33}L^4A_1^2c_1EI_2 - \tfrac{8572}{33}L^4A_1c_1EI_1A_2 + \tfrac{1}{4}L^8c_1^2A_2^2 - \tfrac{8572}{33}L^4c_1A_2A_1EI_2 + \tfrac{8572}{33}L^4c_1A_2^2EI_1 + 67\,474A_1^2E^2I_2^2 - 134\,950A_1E^2I_2I_1A_2 + 67\,474E^2I_1^2A_2^2)^{1/2}]/L^4\rho A_1A_2$$

$$\overline{\omega}_{2,1}^2 = [\tfrac{1}{2}L^4A_1c_1 + \tfrac{1}{2}L^4c_1A_2 + \tfrac{8572}{33}A_1EI_2 + \tfrac{8572}{33}EI_1A_2 + (\tfrac{1}{4}L^8A_1^2c_1^2 + \tfrac{1}{2}L^8A_1c_1^2A_2 + \tfrac{8572}{33}L^4A_1^2c_1EI_2 - \tfrac{8572}{33}L^4A_1c_1EI_1A_2 + \tfrac{1}{4}L^8c_1^2A_2^2 - \tfrac{8572}{33}L^4c_1A_2A_1EI_2 + \tfrac{8572}{33}L^4c_1A_2^2EI_1 + 67\,474A_1^2E^2I_2^2 - 134\,950A_1E^2I_2I_1A_2 + 67\,474E^2I_1^2A_2^2)^{1/2}]/L^4\rho A_1A_2 \tag{30}$$

Numerical evaluation for $L/d = 10$ results in

$$\overline{\omega}_{1,1} = 1.0798636\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.8145735\,\text{THz} \tag{31}$$

Xu et al. [4] reported following results of the exact evaluation of natural frequencies for clamped–clamped DWCNT:

$$\omega_{1,1} = 1.06\,\text{THz}, \quad \omega_{2,1} = 7.75\,\text{THz} \tag{32}$$

The percentagewise difference in estimating $\omega_{1,1}$ is 1.88 percent, whereas its counterpart for $\omega_{2,1}$ is 0.83 percent.

Now we utilize yet another Duncan [15] polynomial that is appropriate for the beam that is clamped at both ends

$$\varphi^{(5)} = \xi^4 - 2\xi^3 + \xi^2 \tag{33}$$

The equations for D_1 and D_2 are obtained as

$$(L^4\rho A_1\overline{\omega}^2 - L^4c_1 - 504EI_1)D_1 + L^4c_1D_2 = 0$$

$$L^4c_1D_1 + (L^4\rho A_2\overline{\omega}^2 - L^4c_1 - 504EI_2)D_2 = 0 \tag{34}$$

The frequency equation reads

$$L^8\rho^2A_1A_2\overline{\omega}^4 + (-504L^4A_1EI_2 - L^8A_1c_1 - 504EI_1L^4A_2 - L^8c_1A_2)\rho\overline{\omega}^2 + 504L^4c_1EI_2 + 504EI_1L^4c_1 + 254\,016E^2I_1I_2 = 0 \tag{35}$$

with roots

$$\begin{aligned}\overline{\omega}_{1,1}^2 = [&L^4A_1c_1 + L^4c_1A_2 + 504A_1EI_2 + 504EI_1A_2 - (L^8A_1^2c_1^2 + 2L^8A_1c_1^2A_2 + 1008L^4A_1^2c_1EI_2 \\ &- 1008L^4A_1c_1EI_1A_2 + L^8c_1^2A_2^2 - 1008L^4c_1A_2A_1EI_2 + 1008L^4c_1A_2^2EI_1 + 254\,016A_1^2E^2I_2^2 \\ &- 508\,032A_1E^2I_2I_1A_2 + 254\,016E^2I_1^2A_2^2)^{1/2}]/2L^4\rho A_1A_2\end{aligned}$$

$$\begin{aligned}\overline{\omega}_{2,1}^2 = [&L^4A_1c_1 + L^4c_1A_2 + 504A_1EI_2 + 504EI_1A_2 + (L^8A_1^2c_1^2 + 2L^8A_1c_1^2A_2 + 1008L^4A_1^2c_1EI_2 \\ &- 1008L^4A_1c_1EI_1A_2 + L^8c_1^2A_2^2 - 1008L^4c_1A_2A_1EI_2 + 1008L^4c_1A_2^2EI_1 + 254\,016A_1^2E^2I_2^2 \\ &- 508\,032A_1E^2I_2I_1A_2 + 254\,016E^2I_1^2A_2^2)^{1/2}]/2L^4\rho A_1A_2 \qquad (36)\end{aligned}$$

Numerical evaluation for $L/d = 10$ yields

$$\overline{\omega}_{1,1} = 1.0636758\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.8130086\,\text{THz} \qquad (37)$$

Comparison between the estimations in $\overline{\omega}_{1,1}$ by using the coordinate functions $\varphi^{(5)}$ as in Eq. (33) shows that the Duncan polynomial as the coordinate functions yields the lower estimate than the Filonenko–Borodich expression. Thus, expressions in (36) are preferable to those in Eq. (30). For $\omega_{2,1}$ Xu et al.'s [4] result in 7.75 THz which is only 0.77 percent lower than the value obtained by the present analysis. The advantage of the current analysis is that the explicit expressions in (34) are obtained herein that allow quick evaluation, albeit approximate, of the fundamental frequency.

7. Clamped–clamped DWCNT: Petrov–Galerkin method

To contrast the results furnished by the Bubnov–Galerkin method, we compare them with the evaluation by the Petrov–Galerkin method. We employ the substitution function $\varphi^{(5)} = \xi^4 - 2\xi^3 + \xi^2$ and the multiplicative function $\psi = 1 - \cos(2\pi\xi)$.

The Petrov–Galekin procedure result in the equations for D_1 and D_2:

$$(L^4\pi^4\rho A_1\overline{\omega}^2 + 45L^4\rho A_1\overline{\omega}^2 - 45L^4c_1 - 720\pi^4EI_1 - L^4c_1\pi^4)D_1 + (45L^4c_1 + L^4c_1\pi^4)D_2 = 0$$

$$(45L^4c_1 + L^4c_1\pi^4)D_1 + (L^4\pi^4\rho A_2\overline{\omega}^2 + 45L^4\rho A_2\overline{\omega}^2 - 45L^4c_1 - 720\pi^4EI_2 - L^4c_1\pi^4)D_2 = 0 \qquad (38)$$

The frequency equation reads

$$\begin{aligned}&20\,280L^8\rho^2A_1A_2\overline{\omega}^4 + (-9\,987\,000L^4A_1EI_2 - 20\,280L^8A_1c_1 - 9\,987\,000EI_1L^4A_2 - 20\,280L^8c_1A_2)\rho\overline{\omega}^2 \\ &+ 9\,987\,000L^4c_1EI_2 + 9\,987\,000EI_1L^4c_1 + 4\,918\,900\,000E^2I_1I_2 = 0 \qquad (39)\end{aligned}$$

with roots

$$\begin{aligned}\overline{\omega}_{1,1}^2 = [&\tfrac{1}{2}L^4A_1c_1 + \tfrac{1}{2}L^4c_1A_2 + \tfrac{9111}{37}A_1EI_2 + \tfrac{9111}{37}EI_1A_2 - (\tfrac{1}{4}L^8A_1^2c_1^2 + \tfrac{1}{2}L^8A_1c_1^2A_2 + \tfrac{9111}{37}L^4A_1^2c_1EI_2 \\ &- \tfrac{9111}{37}L^4A_1c_1EI_1A_2 + \tfrac{1}{4}L^8c_1^2A_2^2 - \tfrac{9111}{37}L^4c_1A_2A_1EI_2 + \tfrac{9111}{37}L^4c_1A_2^2EI_1 + 60\,636A_1^2E^2I_2^2 \\ &- 121\,270A_1E^2I_2I_1A_2 + 60\,636E^2I_1^2A_2^2)^{1/2}]/L^4\rho A_1A_2\end{aligned}$$

$$\begin{aligned}\overline{\omega}_{2,1}^2 = [&\tfrac{1}{2}L^4A_1c_1 + \tfrac{1}{2}L^4c_1A_2 + \tfrac{9111}{37}A_1EI_2 + \tfrac{9111}{37}EI_1A_2 + (\tfrac{1}{4}L^8A_1^2c_1^2 + \tfrac{1}{2}L^8A_1c_1^2A_2 + \tfrac{9111}{37}L^4A_1^2c_1EI_2 \\ &- \tfrac{9111}{37}L^4A_1c_1EI_1A_2 + \tfrac{1}{4}L^8c_1^2A_2^2 - \tfrac{9111}{37}L^4c_1A_2A_1EI_2 + \tfrac{9111}{37}L^4c_1A_2^2EI_1 + 60\,636A_1^2E^2I_2^2 \\ &- 121\,270A_1E^2I_2I_1A_2 + 60\,636E^2I_1^2A_2^2)^{1/2}]/L^4\rho A_1A_2 \qquad (40)\end{aligned}$$

Numerical evaluation for $L/d = 10$ yields

$$\overline{\omega}_{1,1} = 1.0514996\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.8118475\,\text{THz} \qquad (41)$$

Comparison of Eqs. (37) and (41) illustrates that the Petrov–Galerkin method is preferable to the Bubnov–Galerkin method in this case, for it yields the lower estimate for the fundamental frequency.

It is noteworthy that if we interchange the substitution and the multiplicative functions in the Petrov–Galerkin method, i.e. if we use the set $\varphi^{(4)} = 1 - \cos(2\pi\xi)$, $\psi = \xi^4 - 2\xi^3 + \xi^2$ then the same intermediary and final expressions are obtained as above. Hence these will not be reproduced.

8. Simply supported–clamped DWCNT

For this set of boundary conditions, again in the context of Bubnov–Galerkin method, we utilize the following coordinate function:

$$\varphi^{(6)} = 2\xi^4 - 3\xi^3 + \xi \tag{42}$$

The Bubnov–Galerkin procedure yields the equations for D_1 and D_2:

$$(19L^4\rho A_1\overline{\varpi}^2 - 19L^4c_1 - 4536EI_1)D_1 + 19L^4c_1D_2 = 0$$

$$19L^4c_1D_1 + (19L^4\rho A_1\overline{\varpi}^2 - 19L^4c_1 - 4536EI_2)D_2 = 0 \tag{43}$$

The frequency equation reads

$$361L^8\rho^2A_1A_2\bar{\omega}^4 + (-86\,186L^4A_1EI_2 - 361L^8A_1c_1 - 86\,186EI_1L^4A_2 - 361L^8c_1A_2)\rho\overline{\varpi}^2 + 86\,186L^4c_1EI_2 + 86\,186EI_1L^4c_1 + 20\,575\,296E^2I_1I_2 = 0 \tag{44}$$

The natural frequencies squared are

$$\bar{\omega}_{1,1}^2 = [19L^4A_1c_1 + 19L^4c_1A_2 + 4536A_1EI_2 + 4536EI_1A_2 - (361L^8A_1^2c_1^2 + 722L^8A_1c_1^2A_2 + 172\,368L^4A_1^2c_1EI_2 - 172\,368L^4A_1c_1EI_1A_2 + 361L^8c_1^2A_2^2 - 172\,368L^4c_1A_2A_1EI_2 + 172\,368L^4c_1A_2^2EI_1 + 20\,575\,296A_1^2E^2I_2^2 - 41\,150\,592A_1E^2I_2I_1A_2 + 20\,575\,296E^2I_1^2A_2^2)^{1/2}]/38L^4\rho A_1A_2$$

$$\overline{\varpi}_{2,1}^2 = [19L^4A_1c_1 + 19L^4c_1A_2 + 4536A_1EI_2 + 4536EI_1A_2 + (361L^8A_1^2c_1^2 + 722L^8A_1c_1^2A_2 + 172\,368L^4A_1^2c_1EI_2 - 172\,368L^4A_1c_1EI_1A_2 + 361L^8c_1^2A_2^2 - 172\,368L^4c_1A_2A_1EI_2 + 172\,368L^4c_1A_2^2EI_1 + 20\,575\,296A_1^2E^2I_2^2 - 41\,150\,592A_1E^2I_2I_1A_2 + 20\,575\,296E^2I_1^2A_2^2)^{1/2}]/38L^4\rho A_1A_2 \tag{45}$$

For the ratio $L/d = 10$, the numerical evaluation yields

$$\overline{\varpi}_{1,1} = 0.7327673\,\text{THz}, \quad \overline{\varpi}_{2,1} = 7.7862830\,\text{THz} \tag{46}$$

To the best of author's knowledge there is no exact solution available for this case, thus, no comparison with it can be conducted. It makes sense, therefore, to compare Eq. (45) with those obtained via use of *other* coordinate function. For this purpose, we adopt the following polynomial in Bubnov–Galerkin method:

$$\varphi^{(7)} = -5\xi^5 + 16\xi^4 - 14\xi^3 + 3\xi \tag{47}$$

The Bubnov–Galerkin procedure yields the equations for D_1 and D_2:

$$(L^4\rho A_1\overline{\varpi}^2 - L^4c_1 - 264EI_1)D_1 + L^4c_1D_2 = 0$$

$$L^4c_1D_1 + (L^4\rho A_1\overline{\varpi}^2 - L^4c_1 - 264EI_2)D_2 = 0 \tag{48}$$

The frequency equation is obtained as

$$L^8\rho^2A_1A_2\overline{\varpi}^4 + (-264L^4A_1EI_2 - L^8A_1c_1 - 264EI_1L^4A_2 - L^8c_1A_2)\rho\overline{\varpi}^2 + 264L^4c_1EI_2 + 264EI_1L^4c_1 + 69\,696E^2I_1I_2 = 0 \tag{49}$$

The expressions for $\overline{\omega}_{j,i}^2$ are

$$\overline{\omega}_{1,1}^2 = [L^4A_1c_1 + L^4c_1A_2 + 264A_1EI_2 + 264EI_1A_2 - (L^8A_1^2c_1^2 + 2L^8A_1c_1^2A_2 + 528L^4A_1^2c_1EI_2 - 528L^4A_1c_1EI_1A_2 + L^8c_1^2A_2^2 - 528L^4c_1A_2A_1EI_2 + 528L^4c_1A_2^2EI_1 + 69\,696A_1^2E^2I_2^2 - 139\,392A_1E^2I_2I_1A_2 + 69\,696E^2I_1^2A_2^2)^{1/2}]/2L^4\rho A_1A_2$$

$$\overline{\omega}_{2,1}^2 = [L^4A_1c_1 + L^4c_1A_2 + 264A_1EI_2 + 264EI_1A_2 + (L^8A_1^2c_1^2 + 2L^8A_1c_1^2A_2 + 528L^4A_1^2c_1EI_2 - 528L^4A_1c_1EI_1A_2 + L^8c_1^2A_2^2 - 528L^4c_1A_2A_1EI_2 + 528L^4c_1A_2^2EI_1 + 69\,696A_1^2E^2I_2^2 - 139\,392A_1E^2I_2I_1A_2 + 69\,696E^2I_1^2A_2^2)^{1/2}]/2L^4\rho A_1A_2 \quad (50)$$

Numerical values of $\overline{\omega}_{j,1}$ for $L/d = 10$ are, respectively

$$\overline{\omega}_{1,1} = 0.7704939\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.7888259\,\text{THz} \quad (51)$$

Note that the respective values reported in Eqs. (46) and (51) are extremely close.

We also utilize the Petrov–Galerkin method. It is should be emphasized that in two cases, associated with substitution function $\varphi^{(6)}$ or $\varphi^{(7)}$and multiplicative function ψ

$$\varphi^{(6)} = 2\xi^4 - 3\xi^3 + \xi, \quad \psi = -5\xi^5 + 16\xi^4 - 14\xi^3 + 3\xi \quad (52)$$

or

$$\varphi^{(7)} = -5\xi^5 + 16\xi^4 - 14\xi^3 + 3\xi, \quad \psi = 2\xi^4 - 3\xi^3 + \xi \quad (53)$$

we get the same expressions for the natural frequencies. The equations for D_1 and D_2 are

$$(47L^4\rho A_1\overline{\omega}^2 - 47L^4c_1 - 11\,088EI_1)D_1 + 47L^4c_1D_2 = 0$$

$$47L^4c_1D_1 + (47L^4\rho A_1\overline{\omega}^2 - 47L^4c_1 - 11\,088EI_2)D_2 = 0 \quad (54)$$

The frequency equation reads

$$2209L^8\rho^2A_1A_2\overline{\omega}^4 + (-521\,136L^4A_1EI_2 - 2209L^8A_1c_1 - 521\,136EI_1L^4A_2 - 2209L^8c_1A_2)\rho\overline{\omega}^2 + 521\,136L^4c_1EI_2 + 521\,136EI_1L^4c_1 + 122\,943\,744E^2I_1I_2 = 0 \quad (55)$$

with roots

$$\overline{\omega}_{1,1}^2 = [47L^4A_1c_1 + 47L^4c_1A_2 + 11\,088A_1EI_2 + 11\,088EI_1A_2 - (2209L^8A_1^2c_1^2 + 4418L^8A_1c_1^2A_2 + 1\,042\,272L^4A_1^2c_1EI_2 - 1\,042\,272L^4A_1c_1EI_1A_2 + 2209L^8c_1^2A_2^2 - 1\,042\,272L^4c_1A_2A_1EI_2 + 1\,042\,272L^4c_1A_2^2EI_1 + 12\,2943\,744A_1^2E^2I_2^2 - 245\,887\,488A_1E^2I_2I_1A_2 + 122\,943\,744E^2I_1^2A_2^2)^{1/2}]/94L^4\rho A_1A_2$$

$$\overline{\omega}_{2,1}^2 = [47L^4A_1c_1 + 47L^4c_1A_2 + 11\,088A_1EI_2 + 11\,088EI_1A_2 + (2209L^8A_1^2c_1^2 + 4418L^8A_1c_1^2A_2 + 1042272L^4A_1^2c_1EI_2 - 1\,042\,272L^4A_1c_1EI_1A_2 + 2209L^8c_1^2A_2^2 - 1\,042\,272L^4c_1A_2A_1EI_2 + 1\,042\,272L^4c_1A_2^2EI_1 + 122\,943\,744A_1^2E^2I_2^2 - 245\,887\,488A_1E^2I_2I_1A_2 + 122\,943\,744E^2I_1^2A_2^2)^{1/2}]/94L^4\rho A_1A_2 \quad (56)$$

Numerical evaluation for $L/d = 10$ yields

$$\overline{\omega}_{1,1} = 0.72843\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.7850\,\text{THz} \quad (57)$$

Eq. (57) gives the best estimate value for $\omega_{1,1}$ since it is lower than that reported in either Eq. (46) or Eq. (51).

9. Clamped–free DWCNT

In the context Bubnov–Galerkin method we utilize the following Duncan polynomial

$$\varphi^{(8)} = \xi^5 - \tfrac{10}{3}\xi^4 + \tfrac{10}{3}\xi^3 \tag{58}$$

The Bubnov–Galerkin procedure yields the equations for D_1 and D_2

$$(163L^4\rho A_1\overline{\omega}^2 - 163L^4c_1 - 2970EI_1)D_1 + 163L^4c_1D_2 = 0$$

$$163L^4c_1D_1 + (163L^4\rho A_1\overline{\omega}^2 - 163L^4c_1 - 2970EI_2)D_2 = 0 \tag{59}$$

The frequency equation reads

$$\begin{aligned}&26\,569L^8\rho^2A_1A_2\overline{\omega}^4 + (-484\,110L^4A_1EI_2 - 26\,569L^8A_1c_1 - 484\,110EI_1L^4A_2 - 26\,569L^8c_1A_2)\rho\overline{\omega}^2\\ &\quad + 484\,110L^4c_1EI_2 + 484\,110EI_1L^4c_1 + 8\,820\,900E^2I_1I_2 = 0\end{aligned} \tag{60}$$

with roots

$$\begin{aligned}\overline{\omega}_{1,1}^2 = {}&[163L^4A_1c_1 + 163L^4c_1A_2 + 2970A_1EI_2 + 2970EI_1A_2 - (26\,529L^8A_1^2c_1^2 + 53\,138L^8A_1c_1^2A_2\\ &+ 968\,220L^4A_1^2c_1EI_2 - 968\,220L^4A_1c_1EI_1A_2 + 26\,529L^8c_1^2A_2^2 - 968\,220L^4c_1A_2A_1EI_2\\ &+ 968\,220L^4c_1A_2^2EI_1 + 8\,820\,900A_1^2E^2I_2^2 - 17\,641\,800A_1E^2I_2I_1A_2\\ &+ 8\,820\,900E^2I_1^2A_2^2)^{1/2}]/326L^4\rho A_1A_2\end{aligned}$$

$$\begin{aligned}\overline{\omega}_{2,1}^2 = {}&[163L^4A_1c_1 + 163L^4c_1A_2 + 2970A_1EI_2 + 2970EI_1A_2 + (26\,529L^8A_1^2c_1^2 + 53\,138L^8A_1c_1^2A_2\\ &+ 968220L^4A_1^2c_1EI_2 - 968220L^4A_1c_1EI_1A_2 + 26\,529L^8c_1^2A_2^2 - 968\,220L^4c_1A_2A_1EI_2\\ &+ 968\,220L^4c_1A_2^2EI_1 + 8\,820\,900A_1^2E^2I_2^2 - 17\,641\,800A_1E^2I_2I_1A_2\\ &+ 8\,820\,900E^2I_1^2A_2^2)^{1/2}]/326L^4\rho A_1A_2\end{aligned} \tag{61}$$

Numerical evaluation yields

$$\overline{\omega}_{1,1} = 0.20260\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.76411\,\text{THz} \tag{62}$$

Within the Bubnov–Galerkin method we also employ the different expression for the coordinate function, namely,

$$\varphi^{(9)} = \xi^4 - 4\xi^3 + 6\xi^2 \tag{63}$$

The Bubnov–Galerkin procedure yields the following equations for D_1 and D_2:

$$(13L^4\rho A_1\overline{\omega}^2 - 13L^4c_1 - 162EI_1)D_1 + 13L^4c_1D_2 = 0$$

$$13L^4c_1D_1 + (13L^4\rho A_1\overline{\omega}^2 - 13L^4c_1 - 162EI_2)D_2 = 0 \tag{64}$$

The frequency equation reads

$$\begin{aligned}&169L^8\rho^2A_1A_2\overline{\omega}^4 + (-2106L^4A_1EI_2 - 169L^8A_1c_1 - 2106EI_1L^4A_2 - 169L^8c_1A_2)\rho\overline{\omega}^2\\ &\quad + 2106L^4c_1EI_2 + 2106EI_1L^4c_1 + 26\,244E^2I_1I_2 = 0\end{aligned} \tag{65}$$

The roots are

$$\begin{aligned}\bar{\omega}_{1,1}^2 = {}&[13L^4A_1c_1 + 13L^4c_1A_2 + 162A_1EI_2 + 162EI_1A_2 - (169L^8A_1^2c_1^2 + 338L^8A_1c_1^2A_2\\ &+ 4212L^4A_1^2c_1EI_2 - 4212L^4A_1c_1EI_1A_2 + 169L^8c_1^2A_2^2 - 4212L^4c_1A_2A_1EI_2\\ &+ 4212L^4c_1A_2^2EI_1 + 26\,244A_1^2E^2I_2^2 - 52\,488A_1E^2I_2I_1A_2\\ &+ 26\,244E^2I_1^2A_2^2)^{1/2}]/26L^4\rho A_1A_2\end{aligned}$$

$$\begin{aligned}\overline{\omega}_{2,1}^2 = [&13L^4A_1c_1 + 13L^4c_1A_2 + 162A_1EI_2 + 162EI_1A_2 + (169L^8A_1^2c_1^2 + 338L^8A_1c_1^2A_2 \\ &+ 4212L^4A_1^2c_1EI_2 - 4212L^4A_1c_1EI_1A_2 + 169L^8c_1^2A_2^2 - 4212L^4c_1A_2A_1EI_2 \\ &+ 4212L^4c_1A_2^2EI_1 + 26\,244A_1^2E^2I_2^2 - 52\,488A_1E^2I_2I_1A_2 \\ &+ 26\,244E^2I_1^2A_2^2)^{1/2}]/26L^4\rho A_1A_2\end{aligned} \tag{66}$$

From Eq. (66) for $L/d = 10$ we obtain

$$\overline{\omega}_{1,1} = 0.1676\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.76353\,\text{THz} \tag{67}$$

As is seen, $\overline{\omega}_{1,1}$ here is lower than that in Eq. (62). Thus, Eq. (66) is preferable to Eq. (61). Also, comparison with values, reported for this case by Xu et al. [3] shows that the present value for $\overline{\omega}_{1,1}$ differs from his exact solution $\omega_{1,1} = 0.17$ THz by 1.41 percent. For $\overline{\omega}_{2,1}$ Xu et al. [4] reports $\omega_{2,1} = 7.7$ THz the difference with our solution being 0.83 percent.

It ought be emphasized that in the Petrov–Galerkin method, use of either sets

$$\varphi^{(9)} = \xi^4 - 4\xi^3 + 6\xi^2, \quad \psi = \xi^5 - \frac{10}{3}\xi^4 + \frac{10}{3}\xi^3 \tag{68}$$

or

$$\varphi^{(8)} = \xi^5 - \frac{10}{3}\xi^4 + \frac{10}{3}\xi^3, \quad \psi = \xi^4 - 4\xi^3 + 6\xi^2 \tag{69}$$

yields the same expressions. The equations for D_1 and D_2 are

$$\begin{aligned}&(163L^4\rho A_1\overline{\omega}^2 - 163L^4c_1 - 1890 - EI_1)D_1 + 163L^4c_1D_2 = 0 \\ &163L^4c_1D_1 + (163L^4\rho A_1\overline{\omega}^2 - 163L^4c_1 - 1890EI_2)D_2 = 0\end{aligned} \tag{70}$$

The frequency equation reads

$$\begin{aligned}&26\,569L^8\rho^2A_1A_2\overline{\omega}^4 + (-308\,070L^4A_1EI_2 - 26\,569L^8A_1c_1 - 308\,070EI_1L^4A_2 - 26\,569L^8c_1A_2)\rho\overline{\omega}^2 \\ &\quad + 308\,070L^4c_1EI_2 + 308\,070EI_1L^4c_1 + 3\,572\,100E^2I_1I_2 = 0\end{aligned} \tag{71}$$

with roots

$$\begin{aligned}\overline{\omega}_{1,1}^2 = [&163L^4A_1c_1 + 163L^4c_1A_2 + 1890A_1EI_2 + 1890EI_1A_2 - (26\,569L^8A_1^2c_1^2 + 53\,138L^8A_1c_1^2A_2 \\ &+ 616\,140L^4A_1^2c_1EI_2 - 616\,140L^4A_1c_1EI_1A_2 + 26\,569L^8c_1^2A_2^2 - 616\,140L^4c_1A_2A_1EI_2 \\ &+ 616\,140L^4c_1A_2^2EI_1 + 3\,572\,100A_1^2E^2I_2^2 - 7\,144\,200A_1E^2I_2I_1A_2 \\ &+ 357\,210\,044E^2I_1^2A_2^2)^{1/2}]/326L^4\rho A_1A_2 \\ \overline{\omega}_{2,1}^2 = [&163L^4A_1c_1 + 163L^4c_1A_2 + 1890A_1EI_2 + 1890EI_1A_2 + (26\,569L^8A_1^2c_1^2 + 53\,138L^8A_1c_1^2A_2 \\ &+ 616\,140L^4A_1^2c_1EI_2 - 616\,140L^4A_1c_1EI_1A_2 + 26\,569L^8c_1^2A_2^2 - 616\,140L^4c_1A_2A_1EI_2 \\ &+ 616\,140L^4c_1A_2^2EI_1 + 3\,572\,100A_1^2E^2I_2^2 - 7\,144\,200A_1E^2I_2I_1A_2 \\ &+ 357\,210\,044E^2I_1^2A_2^2)^{1/2}]/326L^4\rho A_1A_2\end{aligned} \tag{72}$$

Numerical evaluation yields

$$\overline{\omega}_{1,1} = 0.16162\,\text{THz}, \quad \overline{\omega}_{2,1} = 7.76344\,\text{THz} \tag{73}$$

Eq. (73) gives the best estimate values for $\overline{\omega}_{1,1}$. Xu et al. [4] report the natural frequency values for the cantilever DWCNTs as $\omega_{1,1} = 0.17$ THz and $\omega_{2,1} = 7.7$ THz. Thus, in this case Petrov–Galerkin method ought to be preferred to the Bubnov–Galerkin method.

10. Comparison with results of Natsuki et al. [17]

Most recently, Natsuki et al. [17] analyzed free vibration characteristics of DWCNT. Also, in the private communications [18] to the presents authors, he informed on results of calculation of natural frequencies. Specifically, Natsuki et al. [17,18] adopt the following formula for the van der Waals interaction coefficient c_1:

$$c_1 = \frac{\pi \varepsilon R_1 R_2 \sigma^6}{\alpha^4}\left[\frac{1001\sigma^6}{3}H^{13} - \frac{1120\sigma^6}{9}H^{17}\right] \tag{74}$$

where

$$H^m = (R_1 + R_2)^{-m}\int_0^{\pi/2} 1/(1 - K\cos^2\theta)^{m/2}\,\mathrm{d}\theta, \quad (m = 7, 13) \tag{75}$$

and

$$K = 4R_1R_2/(R_1 + R_2)^2 \tag{76}$$

where σ and ε are the van der Waals radius and the well depth of the Lennard–Jones potential, respectively, $\alpha = 0.142$ nm is the carbon–carbon bond length, R_1 and R_2 are the inner and outer radius, respectively. Our calculations for $\sigma = 0.34$ nm, $\varepsilon = 2.967$ meV, inner diameter $= d_{\mathrm{in}} = 4.8$ nm and outer diameter $= d_{\mathrm{out}} = 5.5$ nm yield $c = 1.474825922044788 \times 10^{11}$ whereas Natsuki [18] informs that his value is 1.50×10^{11}, showing an excellent comparison.

According to Natsuki [18] the first natural frequency for $L = 10$ nm equals 4.04 THz which correlates well with our exact and approximate solutions for simply supported DWCNT. Substituting Natsuki et al.'s [17] values in Eq. (5), we get an exact value for the first natural frequency $\omega_{1,1} = 4.032855669025$ THz which differs from Natsuki's [18] value by only 0.17 percent. The Bubnov–Galerkin method, Eq. (16), gives an approximate value $\omega_{1,1} = 4.03570745028916$ THz which, remarkably, is also very close to Natsuki's value with only 0.099 percent percentagewise difference and 0.07 percent from our exact value.

11. Conclusion

In this study we utilize two approximate methods, namely, those of Bubnov–Galerkin, and Petrov–Galerkin to derive explicit expressions for the double-walled carbon nanotubes under various boundary conditions. The attractiveness of the derived results lies in their simplicity. The expressions are not more complicated that the exact analytical formulas for the double-walled carbon nanotubes, that are simply supported at both end. In the cases where the exact solutions are available our solutions differ by less than two percent, the minimum percentagewise being 0.003 percent. Moreover, to the best of authors' knowledge there are no prior approximate solutions reported in the literature before this study. Thus, availability of present approximate solutions provides a possibility of a quick and effective evaluation of the natural frequency estimates.

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